CSE 125
Discrete Mathematics

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Set

- An **unordered** collection of objects, called **elements** or **members** of the set
- a $\in$ A denotes that a is an element of the set A
- a $\notin$ A denotes that a is not an element of the set A

Sets

- N = {0, 1, 2, 3, . . .}, the set of natural numbers
- Z = {. . . , -2, -1, 0, 1, 2, . . .}, the set of integers
- Z+ = {1, 2, 3, . . .}, the set of positive integers
- Q = {p/q | p $\in$ Z, q $\in$ Z, and q = 0}, the set of rational numbers

Sets

- R, the set of real numbers
- R+, the set of positive real numbers
- C, the set of complex numbers

Example

- The set V of all vowels in the English alphabet is V = {a, e, i, o, u}.
- The set {N, Z, Q, R} is a set containing four elements, N, the set of natural numbers; Z, the set of integers, Q, the set of rational numbers and R, the set of real numbers.

Equal Sets

- Two sets are equal if and only if they have the same elements.
- The set with no elements is called the empty set, or null set, denoted by $\varnothing$, $\{\}$.

Singleton Set

- A set with one element is called a singleton set

  $\{\varnothing\}$ is a singleton set.

Venn Diagram

- Sets can be represented graphically using Venn diagrams.
- The universal set U, containing all the objects under consideration, is represented by a rectangle.
- Circles or other geometrical figures are used to represent sets.

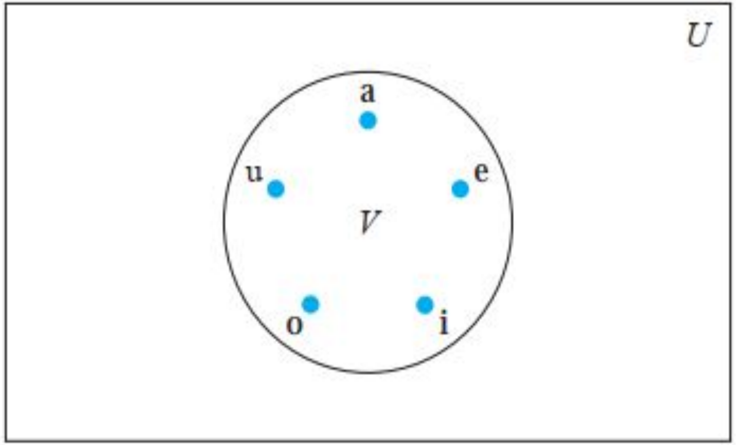


FIGURE 1 **Venn Diagram for the Set of Vowels.**

Subsets

- The set A is a subset of B if and only if every element of A is also an element of B.
- $A \subseteq B$ to indicate that A is a subset of the set B.

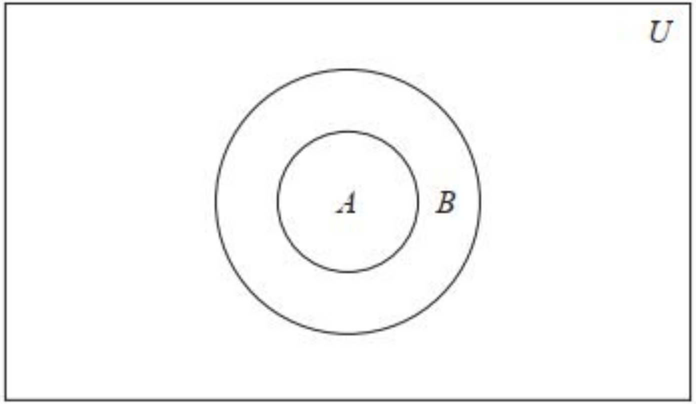


FIGURE 2 **Venn Diagram Showing that A Is a Subset of B.**

Cardinality

- If there are exactly n distinct elements in S where n is a nonnegative integer,

  S is a finite set and n is the cardinality of S. The cardinality of S is denoted by |S|

Example

- Let S be the set of letters in the English alphabet. Then $|S| = 26$
- Null set has no elements, it follows that $|\varnothing| = 0$

Power Set

- The power set of given set, S is the set of all subsets of the set S
- The power set of S is denoted by P(S)

Cartesian Products

- $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$
- $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$
- A relation from A to B is just a subset of $A \times B$

Problem

- What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?
- The Cartesian product $A \times B$ is

  $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

Set Operations

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

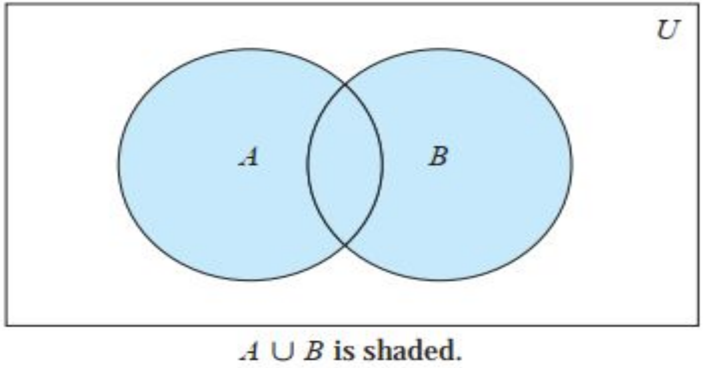


FIGURE 1 **Venn Diagram of the Union of A and B.**

Union

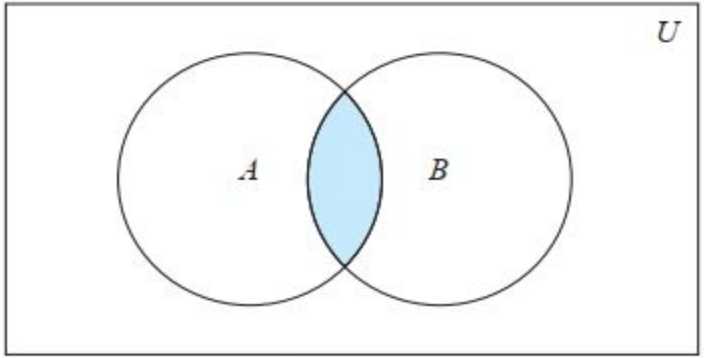
A = {1, 3, 5} B = {1, 2, 3}

A U B ={1, 3, 5} ∪ {1, 2, 3}

= {1, 2, 3, 5}

Intersection

$$A \cap B = \{x \mid x \in A \land x \in B\}$$



$A \cap B$ is shaded.

FIGURE 2 Venn Diagram of the Intersection of A and B.

Intersection

$A = \{1, 3, 5\}$ $B = \{1, 2, 3\}$

$$A \cap B = \{1, 3, 5\} \cap \{1, 2, 3\}$$
$$= \{1, 3\}$$

Difference

$A - B$ or $A \setminus B = \{x \mid x \in A \land x \notin B\}$

The difference of A and B is also called the complement of B with respect to A

Difference

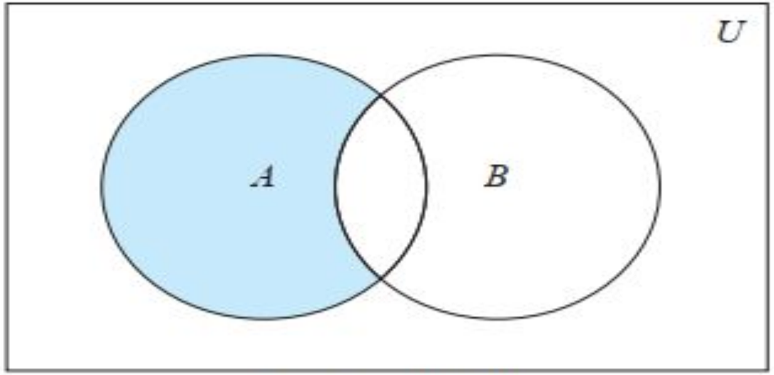
A = {1, 3, 5} B = {1, 2, 3}

A - B = {1, 3, 5} - {1, 2, 3}

= {5}

Again, B - A = {1, 2, 3} - {1, 3, 5}

= {2}

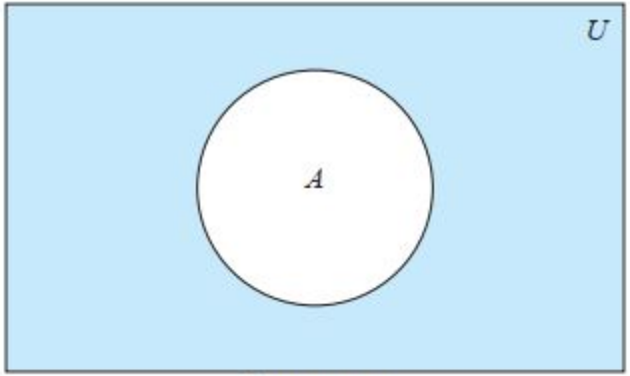


$A - B$ is shaded.

FIGURE 3 Venn Diagram for the Difference of A and B.

Compliment

$A' = \{x \in U \mid x / \in A\}$



$\bar{A}$ is shaded.

FIGURE 4 **Venn Diagram for the Complement of the Set *A*.**

Disjoint

- Two sets are called disjoint if their intersection is the empty set

  $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$

  $A \cap B = \varnothing$,

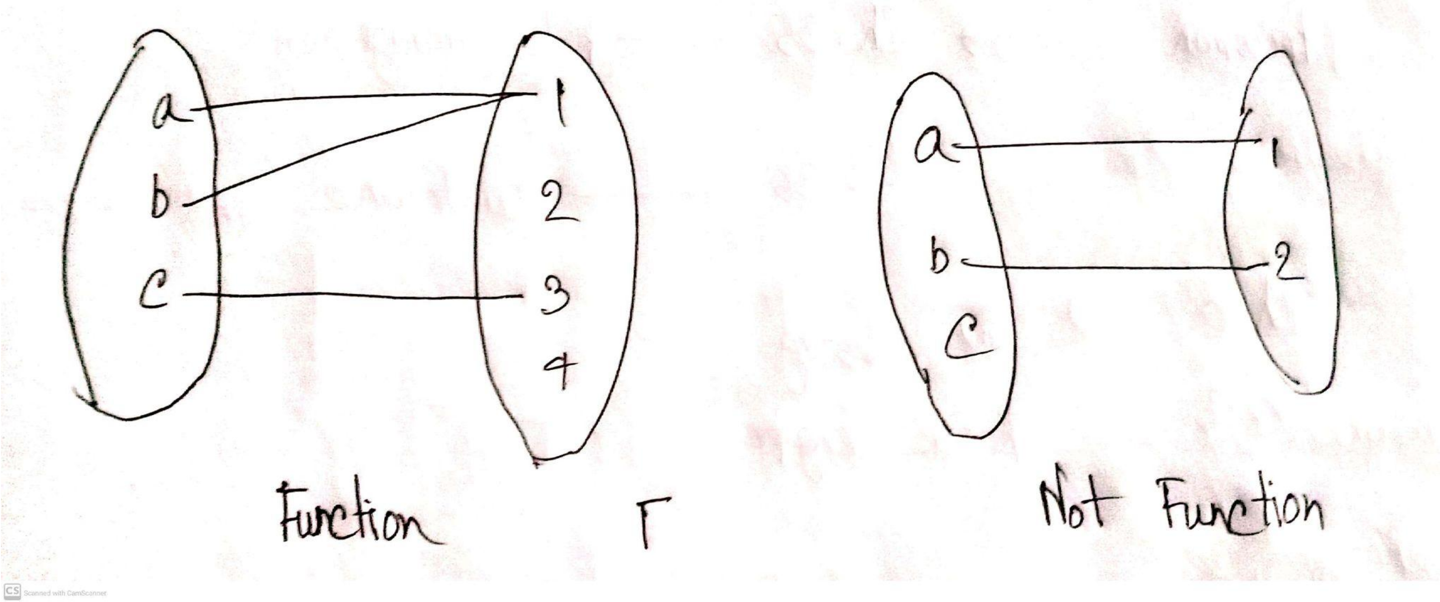
  A and B are disjoint.

Functions

- A function f from non-empty sets, **A** to **B** is an assignment of exactly one element of B to each element of A
- We write f (a) = b if b is the unique element of B assigned by the function f to the element a of A
- If f is a function from A to B, we write $f : A \to B$

A Function

```
int multiply(int a) {
    int ans = a*2;
    return ans;
}
```


a
b
c
1
2
3
4
Function
a
b
c
1
2
Not Function

Functions

- Functions are sometimes also called mappings or transformations
- A relation from A to B that contains one, and only one, ordered pair (a, b) for every element $a \in A$, defines a function f from A to B

Some Terms About Functions

$f : A \rightarrow B$

- Domain: A
- Co - domain: B
- Range: All images of elements of A

Equal Function

Two functions are equal when

- They have the same domain
- Have the same codomain
- Map each element of their common domain to the same element in their common codomain

Problem

Suppose that each student in a discrete mathematics class is assigned a letter grade from the set $\{A, B, C, D, F\}$. And suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens.

What are the domain, codomain, and range of the function that assigns grades to students described above?

Solution

Let G be the function that assigns a grade to a student in our discrete mathematics class.

Note that G(Adams) = A, for instance.

Solution

- The domain of G is the set {Adams, Chou, Goodfriend,Rodriguez, Stevens},
- and the codomain is the set {A, B, C, D, F}.
- The range of G is the set {A, B, C, F}, because each grade except D is assigned to some student.

Problem

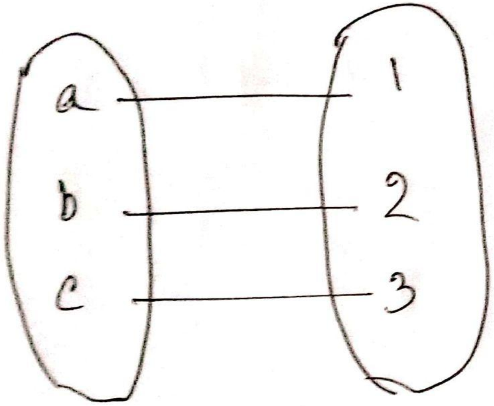
Let $f : Z \to Z$ assign the square of an integer to this integer.

- Then, $f(x) = x^2$, where the domain of f is the set of all integers
- The codomain of f is the set of all integers,
- The range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, . . . \}$

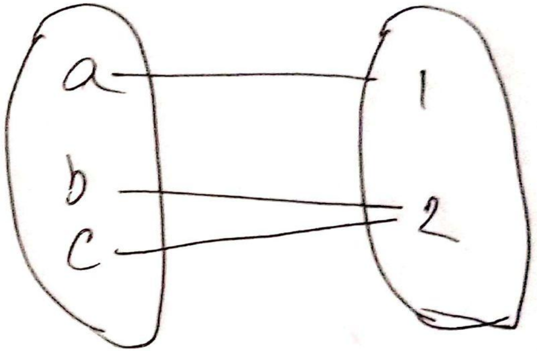
One-to-one Function

A function f is said to be one-to-one, or an injunction, if and only if

- f (a) = f (b) implies that a = b for all a and b in the domain of f
- A function is said to be injective if it is one-to-one



One-to-One



Not One-to-one

One-to-one Function

$\forall a \forall b (f(a) = f(b) \rightarrow a = b)$

or equivalently $\forall a \forall b (a = b \rightarrow f(a) = f(b))$

Problem

Determine whether the function f from {a, b, c, d} to {1, 2, 3, 4, 5} with

$f(a) = 4$, $f(b) = 5$,

$f(c) = 1$, and $f(d) = 3$ is one-to-one

Solution

The function f is one-to-one because f takes on different values at the four elements of its domain.

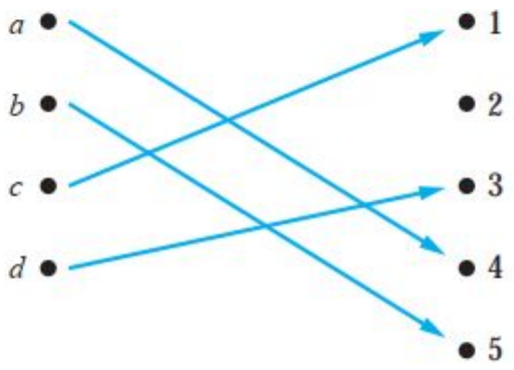


FIGURE 3 **A One-to-One Function.**

Problem

Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution

The function $f(x) = x^2$ is not one-to-one because,

for instance, $f(1) = f(-1) = 1$,

but $1 \neq -1$.

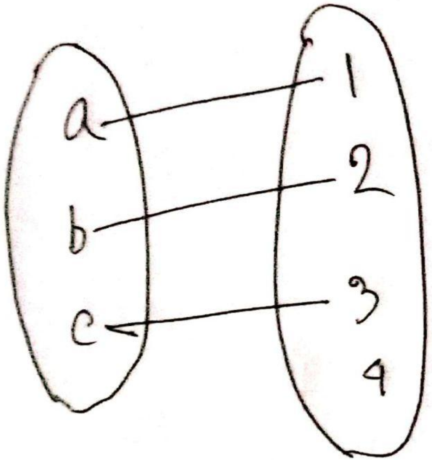
Note that the function $f(x) = x^2$ with its domain restricted to $Z+$ is one-to-one.

Onto Function

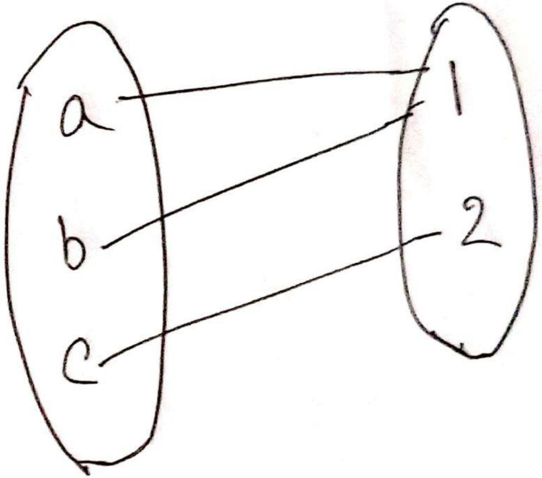
- A function f from A to B is called onto, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.
- A function f is called surjective if it is onto.

Onto Function

A function f is onto if $\forall y \exists x(f(x) = y)$, where the domain for x is the domain of the function and the domain for y is the codomain of the function.



Not Onto



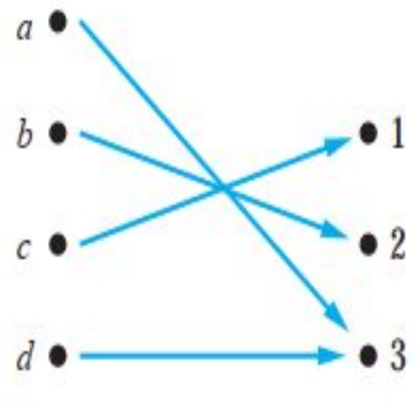
Onto

Problem

Let f be the function from {a, b, c, d} to {1, 2, 3} defined by f (a) = 3, f (b) = 2, f (c) = 1, and f (d) = 3. Is f an onto function?

Solution

- Because all three elements of the codomain ar images of elements in the domain, we see that f is onto.
- Note that if the codomain were {1, 2, 3, 4}, then f would not be onto.

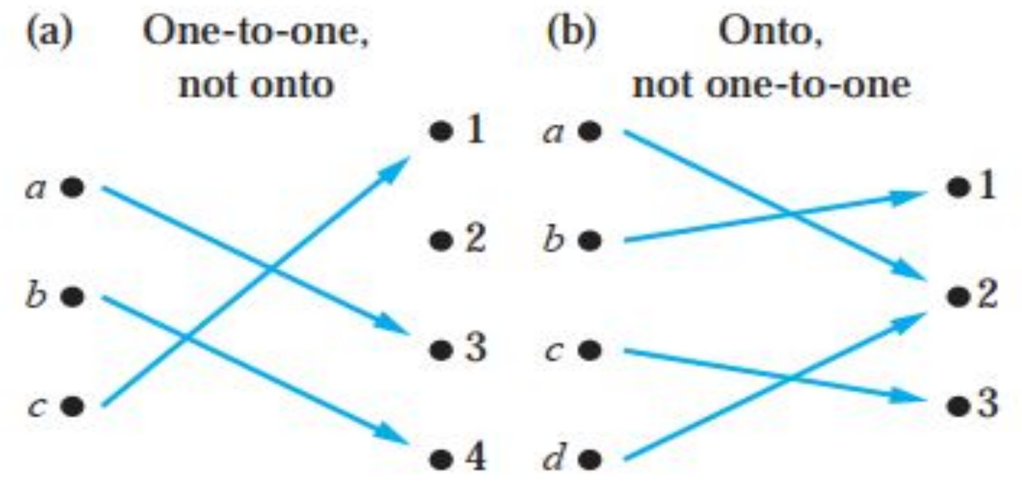


Problem

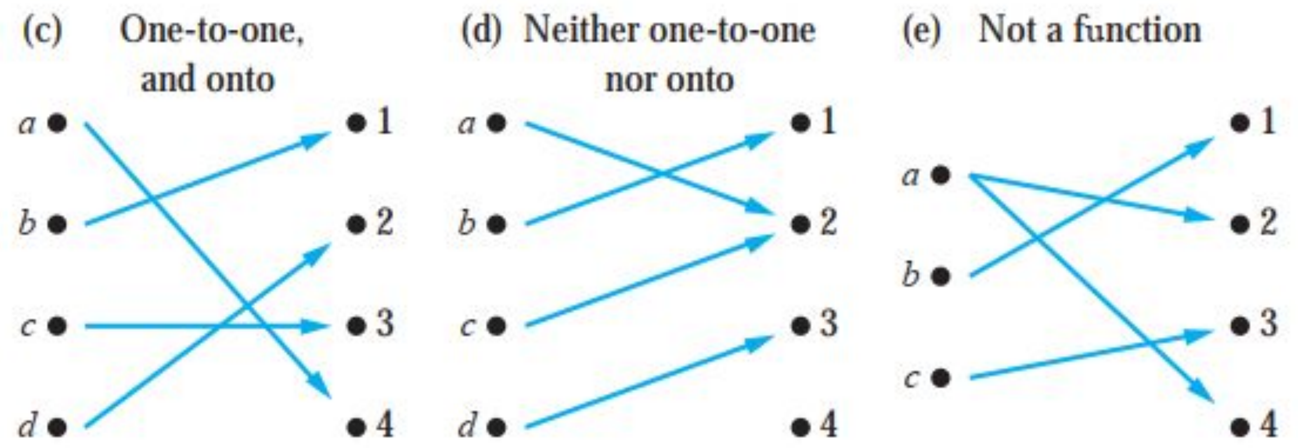
Is the function f (x) = x^2 from the set of integers to the set of integers onto?

Solution: The function f is not onto because there is no integer x with $x^2 = -1$, for instance

Examples of Different Types of Correspondences



Examples of Different Types of Correspondences



More Problems

$f(x) = x - 1$

$f(x) = x^2 + 1$

$f(x) = x^3$

$f(x) = \lceil x/2 \rceil$

$f(x) = 1/x$

$f(x) = \sqrt{x}$

Bijection

- The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto.
- Such a function is bijective.

Problem

- Let f be the function from {a, b, c, d} to {1, 2, 3, 4} with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f a bijection?

Solution

The function f is one-to-one and onto.

- It is one-to-one because no two values in the domain are assigned the same function value.
- It is onto because all four elements of the codomain are images of elements in the domain.

Hence, f is a bijection.

Inverse Function

- The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that f (a) = b
- The inverse function of f is denoted by f −1. Hence, f −1(b) = a when f (a) = b

Inverse Function

- The function f^{-1} and the function $1/f$ are different .
- $1/f$ is the function that assigns to each x in the domain the value $1/f(x)$.
- The latter makes sense only when $f(x)$ is a non-zero real number.

Invertible

- A one-to-one correspondence is called invertible because we can define an inverse of this function.
- A function is not invertible if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

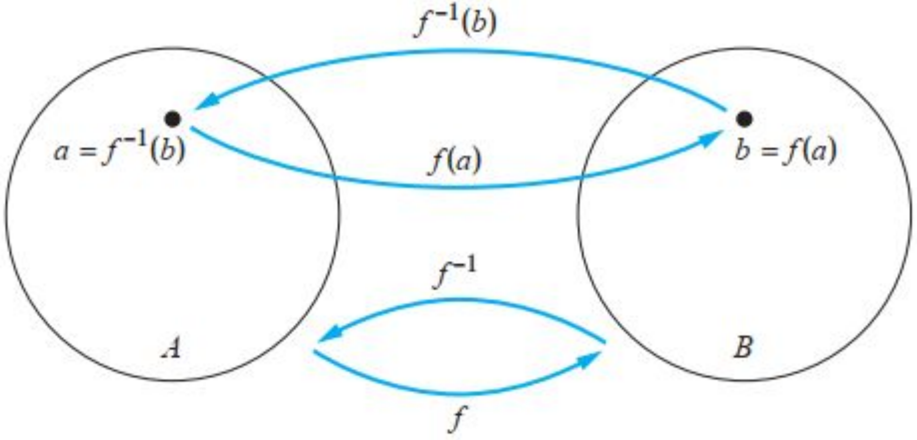


FIGURE 6 **The Function f^{-1} Is the Inverse of Function f.**

Problem

- Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Solution

- The function f is invertible because it is a one-to-one correspondence.
- The inverse function f^{-1} reverses the correspondence given by f , so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Composition Function

- Let g be a function from the set A to the set B and let f be a function from the set B to the set C.

- The composition of the functions f and g, denoted for all a $\in$ A by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.

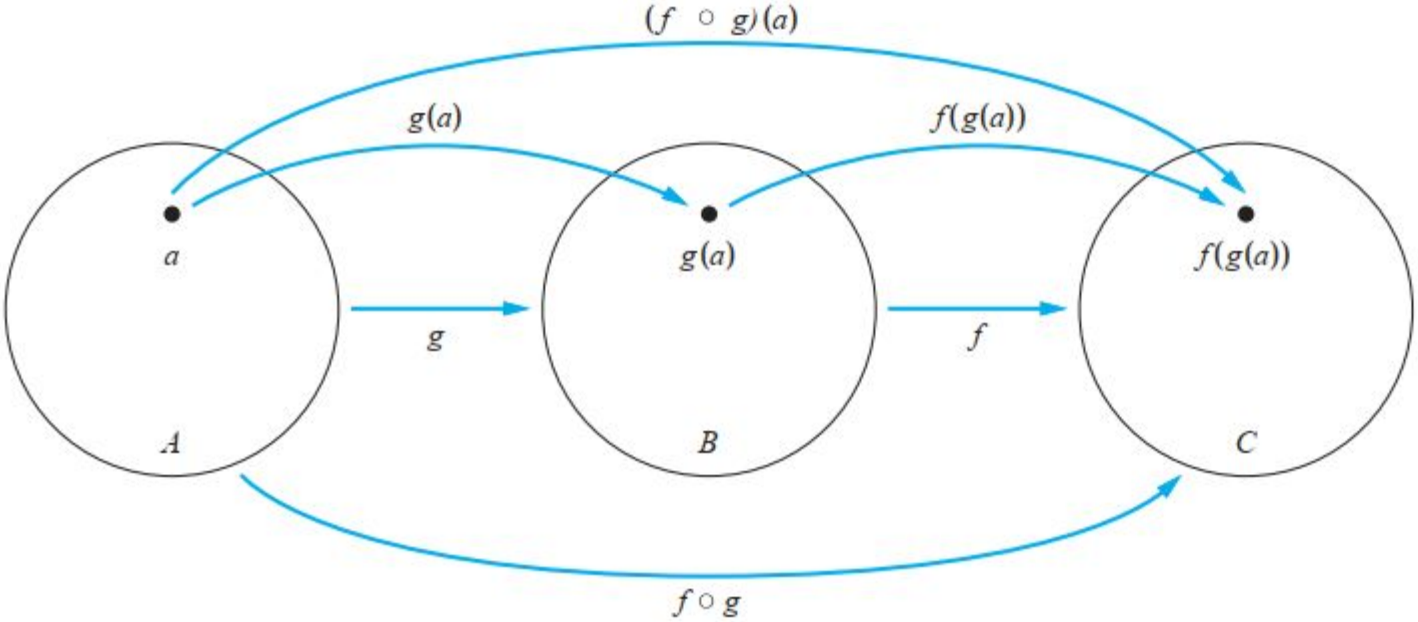


FIGURE 7 **The Composition of the Functions *f* and *g*.**

Problem

- Let g be the function from the set {a, b, c} to itself such that g(a) = b, g(b) = c, and g(c) = a.Let f be the function from the set {a, b, c} to the set {1, 2, 3} such that f (a) = 3, f (b) = 2, and f (c) = 1. What is the composition of f and g, and what is the composition of g and f?

Solution

The composition $f \circ g$ is defined by

- $(f \circ g)(a) = f(g(a)) = f(b) = 2$,
- $(f \circ g)(b) = f(g(b)) = f(c) = 1$, and
- $(f \circ g)(c) = f(g(c)) = f(a) = 3$.

$g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Problem

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

Solution

Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

- $(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$ and
- $(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$.

The Graphs of Functions

- Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$

Problem

Display the graph of the function $f(n) = 2n + 1$ from the set of integers to the set of integers.

Solution

The graph of f is the set of ordered pairs of the form (n, 2n + 1), where n is an integer.The figure is displayed in next slide.

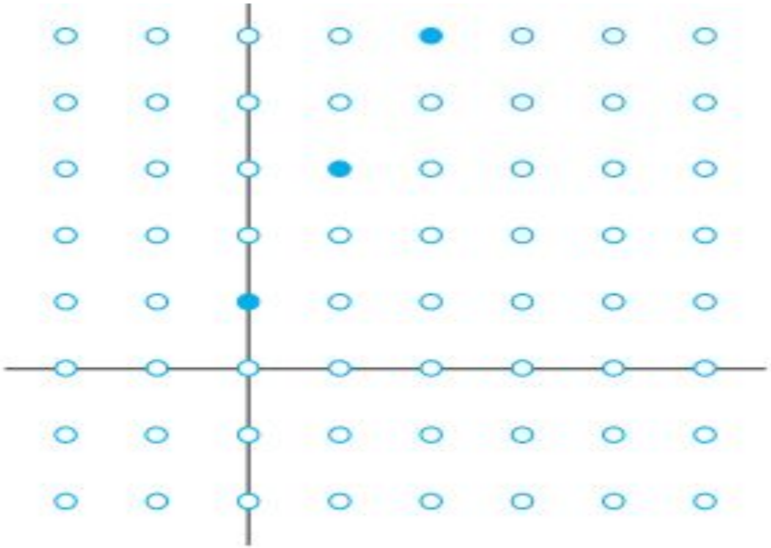


FIGURE 8 The Graph of $f(n) = 2n + 1$ from **Z** to **Z**.

Problem

Display the graph of the function f (x) = x2 from the set of integers to the set of integers.

Solution

The graph of f is the set of ordered pairs of the form $(x, f(x)) = (x, x^2)$, where x is an integer. The graph is displayed in next slide.

FIGURE 9 **The Graph of $f(x) = x^2$ from Z to Z.**

Floor Function

- The floor function assigns to the real number x the largest integer that is less than or equal to x.
- The value of the floor function at x is denoted by $\lfloor x \rfloor$.

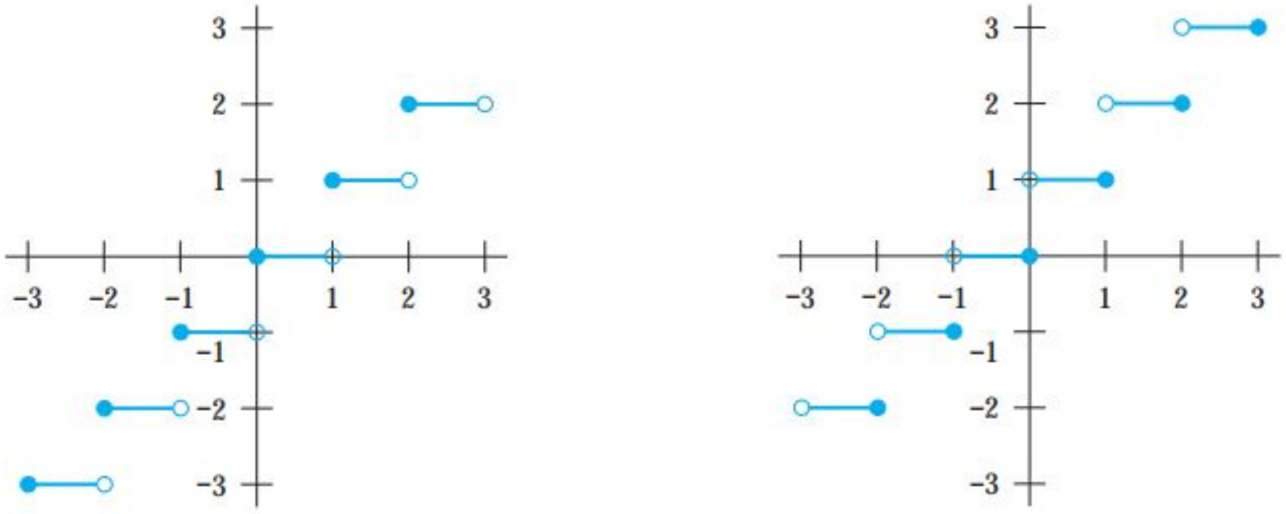
Ceiling Function

- The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x.
- The value of the ceiling function at x is denoted by $\lceil x \rceil$.

Example

These are some values of the floor and ceiling functions:

- $\lfloor 0.5 \rfloor = 0$, $\lceil 0.5 \rceil = 1$,
- $\lfloor -0.5 \rfloor = -1$, $\lceil -0.5 \rceil = 0$,
- $\lfloor 3.1 \rfloor = 3$, $\lceil 3.1 \rceil = 4$,
- $\lfloor 7 \rfloor = 7$, $\lceil 7 \rceil = 7$



(a) $y = \lfloor x \rfloor$ (b) $y = \lceil x \rceil$

FIGURE 10 **Graphs of the (a) Floor and (b) Ceiling Functions.**

Thank You